

The Game of SPROUTS

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CONCEPTS: Mathematical Reasoning, Problem Solving

SKILLS: Applying systematic reasoning skills

CA MATHEMATICS STANDARDS: Gr 1: MG 2.4, MR 3.0; Gr 2: MR 2.0; Gr 3: MR 1.1; Gr 4: MR 1.1, 3.3; Gr 5: MR 1.1; Gr 6: MR 1.1, 1.2, 2.2; Gr 7: MR 1.1, 1.2, 2.4, 2.5, 2.6

GRADES: Kindergarten – middle school

MATERIALS: 8.5" x 11" white paper (or larger, up to poster size paper), colored markers or crayons

Background

Sprouts is a very simple geometry game with sophisticated mathematical strategies. It can be a learning experience for kindergarten students, carefully analyzed for its hidden mathematics by middle school students, and played and enjoyed by all grades in between. Mathematician John Conway, the inventor of many recreational mathematics activities, created Sprouts at Cambridge University in 1967.

Playing the Game

Two players begin by placing two or more large dots on a sheet of paper. It is best if these dots are separated a bit. Players take turns and may not "pass." Each player's turn consists of drawing a line between two dots (or from one dot back to itself), and then adding a new dot somewhere along the new line he/she has just drawn. The new line can be straight or curved, simple or complex. However, players must follow these rules:

1. The line, whether straight or curved, must not touch or cross itself or any other line.
2. The new dot must be placed somewhere on the new line, but not at either endpoint of the line.
3. No dot may have more than three lines connected to it. A line passing through a dot (in and out) counts as two paths, a new dot that is added to a new line automatically has two paths connected to it, and a line from a dot to itself counts as two paths.



The last player to make a legal move by these rules is the winner of the game. If a player makes an illegal move by accident, he or she automatically loses that game. By the same token, if there are still legal moves remaining in a game, but a player is unable to find any of them, the last player who made a legal move is the winner of the game. Here is an example of a 2-dot game of Sprouts.

This is the beginning of a two-dot Sprouts game. The dots can be placed anywhere; they don't have to be vertical.

Player One draws a line (curve) between the two dots and adds the gray dot.

The gray dot is Player Two's first move. The white dot has three lines into it and is completely used up.

This is Player One's second move. He adds the gray dot to the new line.

This is Player Two's second move. She adds the gray dot to the new line.

Although the two white dots are "open," Player One cannot make a move without breaking the rules. Player Two wins the game.

Though the mathematics underlying the strategy of Sprouts is too sophisticated for most primary students, Sprouts is a wonderful geometry game for younger students to hone their observation skills in finding legal moves and planning ahead. Concentration and a steady hand is required to draw each new

line and its new dot. More than two students can play a Sprouts game together, each with his or her own color marker, especially if the paper is larger, such as 12" x 18" construction paper or poster paper off a roll. Save paper by playing it on a white board! Primary students can play using two or three dots to begin a new Sprouts game; however, I do not recommend that younger students use more than three beginning dots because it becomes too difficult for them to see legal moves when the game gets tangled with lines.

Analyzing SPROUTS with Older Students

Upper elementary students can delve deeper into the mathematical strategy of winning Sprouts games. They will discover that "trapped" dots—dots that are not used up, but cannot be reached without breaking the rules of the game—become a critical part of playing Sprouts. Upper elementary students can play with more and more dots to begin a Sprouts game. They will discover on their own when too many dots to start with make a Sprouts game less fun and too hard to play.

Middle school students can take this seemingly simple game to the next level and analyze Sprouts games to determine optimal playing strategies. The mathematical analysis of Sprouts is not trivial, but it can be analyzed algebraically. For example, suppose a Sprouts game begins with an array of n dots, and ends after both players make a total of m moves. Each dot starts out with the potential of three paths into it. Each new move reduces the combined total possible paths into dots by two, e.g., a line that passes clear through an unused dot uses up two of its possible paths, just as a new dot added to a player's new line also starts life with two of its paths used up. At the end of a game there will be $3n - m$ remaining possible paths with dots that are not completely used up. At the end of a game, there must be at least one dot with an open path—the last dot that was added on the last line in the last legal move. Therefore the maximum number of legal moves in a Sprouts game is $3n - 1$.

However, not all Sprouts games go on for exactly $3n - 1$ moves, because it is possible to "trap" a dot so all its paths cannot be used. By simplifying the Sprouts game by using only one dot to begin, students can systematically

analyze all the possible moves and outcomes of a one-dot game, a two-dot game, a three-dot game, and so on. From this analysis, students can determine optimal playing strategies that depend on the number of beginning dots and whether they are Player One or Two in the game. By systematically examining all possible moves, students should be able to show that Player One, when playing with the optimal strategy, will always win a Sprouts game with 3, 4, or 5 dots, while Player Two can win when a Sprouts game has 1, 2, or 6 dots.



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